Table 1 Comparison of optimum cylindrical shells with variable angle spiral, 45° angle spiral, and conventional-type stiffeners

		Best $\bar{t}$, in.			
Design study	$N_{xa}{}^{a}$, p	Variable D_5	$D_5 =$ 45° Ref. 6	Conventional stiffeners Ref. 6	Comments
1[b]	1,000	0.0428 (43.8°)	0.0428	0.0381	
2[b]	10,000	0.199 (45.1°)	0.199	0.168	
3[c,d]	−15,0	0.149 (89.7°)[e]	0.174	0.148[e]	
4[d,f]	−15,0	0.103 (90.0°)	0.116	0.0528	Conventional design is ring-stringer stiffened

[a] N_{xa} is the applied uniform axial compressive load, lb/in., p is the lateral pressure in psi (internal pressure positive), and is zero unless otherwise indicated.
[b] A side constraint on D_5, $20° \leq D_5 \leq 70°$, is used because of the limitations of the local skin buckling equation (optimum D_5 is in parentheses).
[c] The upper limit on D_3 is 1 in.
[d] The skin buckling constraint and the side constraints on D_5 are not applied.
[e] These ring-stiffened designs are essentially similar.
[f] The upper limit on D_3 is 3 in.

a direct comparison of conventionally stiffened, 45° spiral angle, and variable spiral angle, spirally stiffened shells. This comparison is illustrated by Table 1. It may be seen that for this particular set of loading conditions and design parameters spirally stiffened shells are inferior in all cases to the conventional configuration. In fact, for the two studies involving hydrostatic pressure the optimum spiral angles are essentially 90° indicating that the optima are ring-stiffened shells.

The initial synthesis paths of design study 1 converged to an apparent inferior local optimum where $\bar{t} = 0.0440$ and D_5 about 55°. This local optimum seems to result from the conflicting effects of D_5 on the behavior constraints. An attempt to decrease D_5 in this neighborhood will increase the gross buckling load but, on the other hand, will tend to induce local buckling. Such a condition demonstrates the weakness of a design comparison based only on gross buckling behavior.

Design study 1 presented in the table was located using the optimum 45° shell design as a starting point. In design study 2 the design converged to virtually the same design as the 45° spiral angle optimum indicating that, at least for these design parameters, a spiral angle of 45° may be near optimum for the case of a shell under axial compressive load. The existence of other local optima is a distinct possibility and although superior designs might be found one would not normally expect any substantial improvement.

The initial synthesis paths for the design studies (3 and 4) involving hydrostatic pressure, where the range of D_5 was constrained because of the above-mentioned limitation on the local skin buckling equation, resulted in designs that converged on the upper limit of D_5 indicating that the optima might be ring-stiffened. The removal of this constraint, and consequently the local skin buckling constraint, produced the expected ring-stiffened optima presented in the table. A check of these designs shows that they are satisfactory in the panel buckling failure mode.

IV. Conclusions

The few examples given here indicate that spirally stiffened shells may be significantly inferior to shells with conventional-type stiffeners. It is also quite evident that a consideration of local failure modes is vital in any such comparison. These results cannot, however, be considered typical. The superiority of a particular configuration will depend on the shell parameters, loading conditions and side constraints involved in the application, and on the nature of the stiffener cross section.

References

1 Meyer, R. R., "Buckling of 45° Eccentric-Stiffened Waffle Cylinders," *Journal of the Royal Aeronautical Society*, Vol. 71, July 1967, pp. 516–520.

2 Soong, T. C., "Buckling of Cylindrical Shells with Eccentric Spiral-Type Stiffeners," *AIAA Journal*, Vol. 7, No. 1, Jan. 1969, pp. 65–72.

3 Lee, R. L. and Lu, S. Y., "General Instability of Inclined-Stiffened Cylinders Under Bending," *Transactions of the ASME, Ser. E: Journal of Applied Mechanics*, Vol. 36, No. 3, Sept. 1969, pp. 403–407.

4 Meyer, R. R., "Comments on Buckling of Cylindrical Shells with Eccentric Spiral-Type Stiffeners," *AIAA Journal*, Vol. 7, No. 10, Oct. 1969, p. 2047.

5 Soong, T. C., "Reply to R. R. Meyer," *AIAA Journal*, Vol. 7, No. 10, Oct. 1969, pp. 2047–2048.

6 Pappas, M. and Amba-Rao, C. L., "A Direct Search Algorithm for Automated Optimum Structural Design," *AIAA/ASME 11th Structures, Structural Dynamics, and Materials Conference*, AIAA, New York, 1970.

7 Hofmeister, L. D. and Felton, L. P., "Design Examples of Plates with Multiple Rib Sizes," *AIAA/ASME 11th Structures, Structural Dynamics, and Materials Conference*, AIAA, New York, 1970.

8 Durvasula, S., "Flutter of Simply Supported, Parallelogrammic, Flat Panels in Supersonic Flow," *AIAA Journal*, Vol. 5, No. 9, Sept. 1967, pp. 1668–1673.

Large-Amplitude Slow Oscillation of Wedges in Inviscid Hypersonic and Supersonic Flows

W. H. Hui*

University of Southampton, Southampton, England

Nomenclature

a	=	speed of sound, nondimensionalized by $\bar{u}_\infty$
G	=	$\tau[1 - \cos(\theta - \theta_a)] + \sin(\theta - \theta_a)$
H	=	$(\sin\theta - \sin\theta_a)/\cos\theta$
h	=	nondimensional distance of pivot position from the apex
K	=	$M_\infty \sin\beta$, also hypersonic similarity parameter
k	=	$\omega\bar{l}/\bar{u}_\infty$, frequency parameter
$\bar{l}$	=	wedge chord
M	=	Mach number
q_0	=	$(u_0{}^2 + v_0{}^2)^{1/2}$
β	=	quasi-steady shock angle
θ	=	instantaneous incidence of wedge surface
α	=	relative amplitude of oscillation, Eq. (14)
τ	=	$\tan\beta$
δ	=	$\tan\theta$
σ	=	$\tau - \delta$
$(\dot{\ })$	=	d/dt

Subscripts

∞	=	freestream
0	=	quasi-steady value
1	=	first-order quantities
a	=	average
b	=	body surface

SMALL-AMPLITUDE oscillation of wedges in inviscid hypersonic and supersonic flows has been studied by using an amplitude expansion technique,[1] and a solution has been obtained which can be applied to any wedges provided the bow shock is attached. For large-amplitude slow oscillation of a wedge, Kuiken[2] has obtained a solution by per-

Received March 13, 1970.

* Lecturer, Department of Aeronautics and Astronautics. Member AIAA.

turbing about an instantaneous wedge position at any time during its oscillation. However, his solution is valid only for slender wedges in hypersonic flow, and the amplitude of oscillation must be restricted in such a way that at all time during the oscillation the instantaneous incidence of the wedge surface is sufficiently small for the hypersonic small-disturbance approximation to be applicable. The purpose of this paper is to extend Kuiken's results, by combining the techniques in Refs. 1 and 2, to the general case of large-amplitude slow oscillation of any wedge angle in both hypersonic and supersonic flows provided the bow shocks on both sides of the wedge are attached. A simple solution in closed form is obtained which, for the special case of a slender wedge in hypersonic flow, reduces to Kuiken's solution.

Since the flows on the upper and lower surfaces of the wedge are independent, we shall be concerned only with the upper-surface flow. The coordinates to be used are shown in Fig. 1. Physical quantities are denoted with a bar, thus $\bar{p}(\bar{x},\bar{y},\bar{t})$, $\bar{\rho}$, $\bar{u}$, and $\bar{v}$ are, respectively, the pressure, density, and the $\bar{x}$ and $\bar{y}$ components of velocity of the fluid. Nondimensional quantities are introduced as follows:

$$x = \bar{x}/\bar{l},\ y = \bar{y}/\bar{l},\ t = \bar{t}\bar{u}_\infty/\bar{l}$$

$$u = \bar{u}/\bar{u}_\infty,\ v = \bar{v}/\bar{u}_\infty,\ p = \bar{p}/\bar{p}_\infty\gamma M_\infty^2,\ \rho = \bar{\rho}/\bar{\rho}_\infty$$

Consider a slow pitching motion of a wedge with circular frequency ω such that the frequency parameter $k = \omega\bar{l}/\bar{u}_\infty$ is very small and the flow quantities can be sought in the form, e.g.,

$$p(x,y,t) = p_0[\theta(t)] + \sum_{n=1}^{\infty} k^n p_n[x,y,\theta(t)] \quad (1)$$

where θ is the instantaneous incidence of the wedge surface and the quantities with a subscript 0 represent the quasi-steady values which are given functions of θ as below:

$$p_0 = [2\gamma K^2 - (\gamma - 1)]/\gamma(\gamma + 1)M_\infty^2 \quad (2a)$$

$$u_0 = 1 - 2(K^2 - 1)/(\gamma + 1)M_\infty^2 \quad (2b)$$

$$v_0 = 2(K^2 - 1)/(\gamma + 1)M_\infty^2 \tan\beta \quad (2c)$$

$$\rho_0 = (\gamma + 1)K^2/[(\gamma - 1)K^2 + 2] \quad (2d)$$

$$\tan\theta = 2(K^2 - 1)/[M_\infty^2(\gamma + \cos 2\beta) + 2]\tan\beta \quad (2e)$$

Perturbing with respect to k the equations of motion of an inviscid fluid and retaining only terms of order k, we obtain

$$\left(u_0\frac{\partial}{\partial x} + v_0\frac{\partial}{\partial y}\right)\rho_1 + \rho_0\left(\frac{\partial u_1}{\partial x} + \frac{\partial v_1}{\partial y}\right) = \rho_0 g_4(t) \quad (3a)$$

$$\left(u_0\frac{\partial}{\partial x} + v_0\frac{\partial}{\partial y}\right)u_1 + \frac{1}{\rho_0}\frac{\partial p_1}{\partial x} = u_0\, g_2(t) \quad (3b)$$

$$\left(u_0\frac{\partial}{\partial x} + v_0\frac{\partial}{\partial y}\right)v_1 + \frac{1}{\rho_0}\frac{\partial p_1}{\partial y} = u_0 g_3(t) \quad (3c)$$

$$\left(u_0\frac{\partial}{\partial x} + v_0\frac{\partial}{\partial y}\right)(p_1 - a_0^2\rho_1) = \rho_0 a_0[u_0 g_1(t) - a_0 g_4(t)] \quad (3d)$$

where

$$g_1 = -\dot{p}_0/k\rho_0 a_0 u_0,\ g_2 = -\dot{u}_0/ku_0$$
$$g_3 = -\dot{v}_0/ku_0,\ g_4 = -\dot{\rho}_0/k\rho_0 \quad (4)$$

Making a transformation

$$\xi = (u_0 x + v_0 y)/q_0^2,\ \eta = (v_0 x - u_0 y)/q_0^2 \quad (5)$$

Eqs. (3) become

$$(M_0^2 - 1)(\partial^2 p_1/\partial\xi^2) - (\partial^2 p_1/\partial\eta^2) = 0 \quad (6a)$$

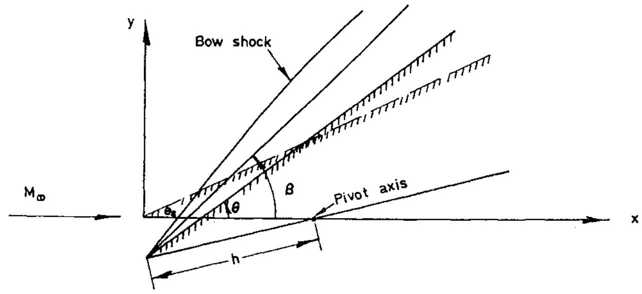

Fig. 1 Wedge showing notation.

$$\frac{\partial u_1}{\partial\xi} + \frac{1}{\rho_0 q_0^2}\left(u_0\frac{\partial p_1}{\partial\xi} + v_0\frac{\partial p_1}{\partial\eta}\right) = u_0 g_2 \quad (6b)$$

$$\frac{\partial v_1}{\partial\xi} + \frac{1}{\rho_0 q_0^2}\left(v_0\frac{\partial p_1}{\partial\xi} - u_0\frac{\partial p_1}{\partial\eta}\right) = u_0 g_3 \quad (6c)$$

$$(\partial/\partial\xi)(p_1 - a_0^2\rho_1) = \rho_0 a_0(u_0 g_1 - a_0 g_4) \quad (6d)$$

The general solution of the first equation of (6) is

$$p_1 = f_1[\xi - (M_0^2 - 1)^{1/2}\eta] + f_2[\xi + (M_0^2 - 1)^{1/2}\eta]$$

where f_1 and f_2 are arbitrary functions. After p_1 is found, u_1, v_1 and ρ_1 can be found from the second, the third, and the fourth equation, respectively, of (6).

To state the boundary conditions, we note that the equation of the wedge surface is

$$y = \delta x - hH(\theta) \quad (7)$$

and the boundary condition on it is, at $y = \delta x - hH$,

$$v_1 - \delta u_1 - \frac{\dot{\theta}}{k}(1 + \delta^2)[x - h(1 - \sin\theta\sin\theta_a)] = 0 \quad (8)$$

The shock is described by

$$y = \tau x - hG + kW_1(x,\theta) \quad (9)$$

where W_1 represents the departure of the shock from its quasi-steady position. The boundary conditions at the bow shock are obtained by perturbing the Rankine-Hugoniot conditions for small value of $k\partial W_1/\partial x$ as follows: at $y = \tau x - hG$,

$$p_1 = \frac{2}{\gamma + 1}\sin 2\beta\cos^2\beta\frac{\partial W_1}{\partial x} \equiv \rho_0 u_0 r_1\frac{\partial W_1}{\partial x} \quad (10a)$$

$$u_1 = -\frac{2}{\gamma + 1}\sin 2\beta\cos^2\beta\frac{\partial W_1}{\partial x} \equiv r_2\frac{\partial W_1}{\partial x} \quad (10b)$$

$$v_1 = \frac{2}{\gamma + 1}\frac{\cos^2\beta(1 + K^2\cos 2\beta)}{K^2}\frac{\partial W_1}{\partial x} \equiv r_3\frac{\partial W_1}{\partial x} \quad (10c)$$

$$\rho_1 = \frac{4}{\gamma + 1}\frac{\rho_0^2}{M_\infty^2}\cot^3\beta\frac{\partial W_1}{\partial x} \equiv r_4\frac{\partial W_1}{\partial x} \quad (10d)$$

The problem is now to solve Eqs. (6) to satisfy the boundary conditions (8) and (10). A solution is found as follows:

$$p_1 = \rho_0 u_0(A_1 x + B_1 y + C_1),\ u_1 = A_2 x + B_2 y + C_2$$
$$v_1 = A_3 x + B_3 y + C_3,\ W_1 = \tfrac{1}{2}A_4 x^2 + B_4 x + \text{const} \quad (11)$$

where the coefficients A_1 through B_4 are

$$A_4 = \frac{(1 + \delta\tau)g + \delta(g_2 + \tau g_3) + (1 + \delta^2)\dot{\theta}/k}{\delta r_1 + (1 + \delta\tau)(r_3 - r)} \quad (12a)$$

$$B_4 = -\frac{h(1 - \sin\theta\sin\theta_a)(1 + \delta^2)\dot{\theta}/k}{r_3 - \delta r_2} \quad (12b)$$

$$A_1 = [r_1 + \tau(r_3 - r)]A_4 - \tau(g_3 + g) \quad (12c)$$

$$A_2 = \frac{\tau}{\sigma}\left[-\left(r_1 + \frac{\delta}{\tau}r_2 + \tau r_3 - \tau r\right)A_4 + (g_2 + \tau g_3 + \tau g)\right] \quad (12d)$$

$$A_3 = [r_3 - (\tau/\sigma)r]A_4 - (\tau/\sigma)g \quad (12e)$$

$$B_1 = (r - r_3)A_4 + (g_3 + g) \quad (12f)$$

$$B_2 = [(r_1 + r_2 + \tau r_3 - \tau r)A_4 - (g_2 + \tau g_3 + \tau g)]/\sigma \quad (12g)$$

$$B_3 = (rA_4 + g)/\sigma \quad (12h)$$

$$C_1 = hGB_1 + r_1B_4 \quad (12i)$$

$$C_2 = hGB_2 + r_2B_4 \quad (12j)$$

$$C_3 = hGB_3 + r_3B_4 \quad (12k)$$

where

$$g = \frac{m\sigma g_1 - \tau g_2 + (m^2\sigma^2 - \tau^2)g_3}{1 + \tau^2 - m^2\sigma^2}$$

$$r = -\frac{(m^2\sigma - \tau)r_1 - \delta r_2 + (m^2\sigma^2 - \tau^2)r_3}{1 + \tau^2 - m^2\sigma^2} \quad (13)$$

The quantities g_1, g_2, and g_3 can be calculated in the following way, e.g.,

$$g_1 = -(1/k\rho_0 a_0 u_0)(\partial p_0/\partial\beta)(\partial\beta/\partial\theta)\dot{\theta}$$

where $\partial p_0/\partial\beta$ and $\partial\beta/\partial\theta$ can be obtained from Eq. (2), and $\dot{\theta}$ can be calculated if the wedge is assumed to oscillate harmonically according to

$$\theta = \theta_a(1 + \alpha \cos kt), \ |\alpha| < 1 \quad (14)$$

It is seen from Eqs. (12–14) that the coefficients A_1 through B_4 and hence the first-order solution are proportional to the amplitude of oscillation. This is the same as in the case of small amplitude oscillations. What is different is, as pointed out in Ref. 2, that for large amplitudes the constants of proportionality depend on the instantaneous (quasi-steady) flow quantities, whereas for small amplitudes they depend on the mean flow quantities. Thus for large-amplitude oscillations the flowfield is highly nonlinear, as expected.

We shall now show that for the special case of hypersonic flow past a slender wedge the present solution reduces to that of Kuiken.[2] A comparison will be given of the pressure at the wedge surface derived by either theory. According to the present theory the pressure at the body surface is given by

$$\frac{\bar{p}_b}{\bar{p}_\infty \gamma M_\infty^2} = p_0 + k\rho_0 u_0[(A_1 + \delta B_1)x + C_1 - hHB_1] \quad (15)$$

For hypersonic flow past a slender wedge, β and θ are all small, and when their quadratic terms are neglected, Eq. (15) can be simplified and put in a form directly comparable with Eq. (54) of Ref. 2. Thus

$$\bar{p}_b/\bar{p}_\infty = \gamma K^2(p_0/\beta^2)[1 + kA(Rx + Sh)] \quad (16)$$

where

$$A = -\frac{\alpha \sin kt}{1 + \alpha \cos kt}\frac{K^2 - 1}{K^2 + 1} = -\frac{\alpha \sin(kt)(K_a^2 - 1)/2K_a}{\left[\left\{\frac{K_a^2 - 1}{2K_a}(1 + \alpha \cos kt)\right\}^2 + 1\right]^{1/2}} \quad (17)$$

$$R = \frac{4\gamma(K^2+1)}{2\gamma K^2 - (\gamma - 1)}\left[1 + \frac{K^2}{K^2+1} \times \frac{(\gamma+1)(K^4-1) - (3K^2+1)\{(\gamma-1)K^2+2\}}{(\gamma+1)(K^4-1) + (3K^2+1)\{(\gamma-1)K^2+2\}}\right]$$

$$S = -4\gamma K^2/[2\gamma K^2 - (\gamma - 1)]$$

Now A and S are identical to the same quantities in Ref. 2, and R gives exactly the same values as those tabulated in Ref. 2. Hence Eq. (16) is identical to Eq. (54) of Ref. 2, and we may conclude that for the special case of hypersonic flow past a slender wedge, the present solution reduces to that of Kuiken. However, the present solution can be used for both hypersonic and supersonic flows past any wedges with attached bow shocks on both sides of the wedge.

References

[1] Hui, W. H., "Stability of Oscillating Wedges and Caret Wings in Hypersonic and Supersonic Flows," *AIAA Journal*, Vol. 7, No. 8, Aug. 1969, pp. 1524–1530.

[2] Kuiken, A. K., "Large-Amplitude Low-Frequency Oscillation of a Slender Wedge in Inviscid Hypersonic Flow," *AIAA Journal*, Vol. 7, No. 9, Sept. 1969, pp. 1767–1774.

Further Experimental Studies of Buckling of Electroformed Conical Shells

R. L. Sendelbeck*
Stanford University, Stanford, Calif.

AND

J. Singer†
Technion, Israel Institute of Technology, Haifa, Israel

In an earlier experimental investigation,[1] the validity of linear theory for buckling of conical shells under hydrostatic pressure was confirmed also for large taper ratios. Correlation of the critical pressures of conical shells with those of equivalent cylindrical shells[2–4] yields a curve (or curves) that have a "hump" at large taper ratios. Since Singer and Bendavid aimed at confirmation of the theory in the "hump" region, all the test specimens had fairly large taper ratios. The results for conical shells of large taper ratio were then correlated with previous tests on other conical shells and cylindrical shells (i.e., shells of zero taper ratio). It may be argued that for a higher degree of certainty, "control" specimens of small taper ratio from the same batch, or same manufacturing technique, should have been included. This motivated the present series of tests, which repeated tests similar to those of Ref. 1 but accompanied them with parallel tests of corresponding specimens of small taper ratio. The results reconfirmed and substantiated the findings of Singer and Bendavid.

The original test apparatus and procedure of Ref. 1 were employed after incorporation of some improvements. For better centering the original gallows (Fig. 3 of Ref. 1) was re-

Received April 6, 1970. This research was supported by the Air Force Office of Scientific Research, OAR, U. S. Air Force under Contract F44620-69-C-009.

* Director, Structures Laboratory, Department of Aeronautics and Astronautics.

† Professor, Department of Aeronautical Engineering. Associate Fellow AIAA.